Positivity properties of canonical bases

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Quantum Groups

- Let $(I, \cdot)$ be a symmetric Cartan datum.
- For $n \in \mathbb{N}$, let $[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$, $[n]_! = \prod_{k=1}^{n} [k]$.
- The usual n^- part of semisimple (and Kac-Moody Lie algebra) is generated f_i subject to the Serre relation: $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$.
- Let $\mathbf{f}$ be the $\mathbb{Q}(v)$-algebra with 1 generated by θ_i for $i \in I$ subject to the quantum Serre relations,

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \theta_i^{(n)} \theta_j \theta_i^{(1-a_{ij}-n)} = 0 \text{ for } i \neq j \text{ in } I,$$

where $\theta_i^{(n)} = \theta_i^n / [n]_!$ for $i \in I, n \in \mathbb{N}$.

Quantum Groups

- The Lie algebra is generated by e_i, f_i and h satisfying
  1. $[h, h'] = 0$;
  2. $[h, e_i] = \alpha_i(h)e_i, [h, f_i] = -\alpha_i(h)f_i$;
  3. $[e_i, f_j] = \delta_{ij}h_i$;
  4. Serre relation for e_i and for f_i.
- Drinfeld-Jimbo quantum group $\mathbf{U}$ is the $\mathbb{Q}(v)$-algebra with 1 generated by E_i, F_i, K_μ for $i \in I, \mu \in Y$ satisfying
  1. $K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in Y$;
  2. $K_\mu E_i = v^{\langle \mu, i' \rangle} E_i K_\mu$ for all $i \in I, \ \mu \in Y$;
  3. $K_\mu F_i = v^{-\langle \mu, i' \rangle} F_i K_\mu$ for all $i \in I, \ \mu \in Y$;
  4. $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{v_i - v_i^{-1}}$;
  5. Quantum Serre relation for E_i and for F_i.

Quantum Groups

- Drinfeld-Jimbo quantum group $\mathbf{U}$ has a triangular decomposition

  $$\mathbf{U} = \mathbf{U}^+ \otimes \mathbf{U}^0 \otimes \mathbf{U}^- = \langle E_i; i \in I \rangle \otimes \langle K_\mu; \mu \in Y \rangle \otimes \langle F_i; i \in I \rangle.$$

- There are algebra isomorphisms $\mathbf{f} \xrightarrow{\cong} \mathbf{U}^\pm, x \mapsto x^\pm$.
- Lusztig's modified quantum group $\dot{\mathbf{U}}$ is a modified form of $\mathbf{U}$ in which $\mathbf{U}^0$ is replaced by a "smaller" one. It is more appropriate than $\mathbf{U}$ for the study of weight modules.

Canonical basis of $\mathbf{f}$

Lusztig introduced the canonical basis $\mathbf{B}$ of $\mathbf{f}$. This basis has remarkable properties:

- It is a $\mathbb{Z}[v, v^{-1}]$-basis of the integral form ${}_{\mathcal{A}}\mathbf{f}$.
- It is fixed by the bar-involution.
- Almost orthogonal property: $(b, b')_{\mathbf{f}} \in \delta_{b,b'} + v^{-1}\mathbb{Z}[[v^{-1}]]$ for $b, b' \in \mathbf{B}$.

  A cornerstone of the theory is its remarkable positivity.

Theorem (Lusztig)

The canonical basis $\mathbf{B}$ of $\mathbf{f}$ has the following positivity:

$$bb' \in \mathbb{N}[v, v^{-1}][\mathbf{B}],\ r(b) \in \mathbb{N}[v, v^{-1}][\mathbf{B} \otimes \mathbf{B}] \textit{ for } b, b' \in \mathbf{B}.$$

Here $r : \mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$ is the comultiplication of $\mathbf{f}$.

Canonical Bases for Simple Modules

- For any weight ζ, let M_ζ be the Verma module.
- For any dominant weight λ, let Λ_λ be the simple highest weight module with highest weight λ and ${}^\omega\Lambda_\lambda$ be the simple lowest weight module with lowest weight $-\lambda$.
- Lusztig proved that $\mathbf{B}(\Lambda_\lambda) = \{b^- \eta_\lambda; b \in \mathbf{B}\} \setminus \{0\}$ is a basis of Λ_λ, and defined it to be the canonical basis of Λ_λ. This basis also has very remarkable properties, including:

Theorem (Lusztig)

The canonical basis $\mathbf{B}(\Lambda_\lambda)$ of Λ_λ has the following positivity:

$$F_i(\mathbf{B}(\Lambda_\lambda)), E_i(\mathbf{B}(\Lambda_\lambda)) \subset \mathbb{N}[v, v^{-1}][\mathbf{B}(\Lambda_\lambda)] \text{ for any } i \in I.$$

The canonical basis $\mathbf{B}({}^\omega\Lambda_\lambda) = \{b^+ \xi_{-\lambda}; b \in \mathbf{B}\} \setminus \{0\}$ of ${}^\omega\Lambda_\lambda$ has similar properties.

The Challenge: Tensor Products

- The naive basis $\mathbf{B}({}^{\omega}\Lambda_{\lambda_1}) \otimes \mathbf{B}(\Lambda_{\lambda_2})$ of ${}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}$ is not compatible with the bar-involution on $\mathbf{U}$ and the action of $\mathbf{U}$.
- Lusztig defined a new involution Ψ on the tensor product ${}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}$ which is compatible with the bar-involution on $\mathbf{U}$ and the action of $\mathbf{U}$. The involution Ψ leads to the correct canonical basis $\mathbf{B}({}^{\omega}\Lambda_{\lambda_1})\diamondsuit\mathbf{B}(\Lambda_{\lambda_2})$ of the tensor product ${}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}$.
- This leads to the canonical basis $\dot{\mathbf{B}}$ of the modified quantum group $\dot{\mathbf{U}}$.
- Lusztig and later Bao-Wang generalized above result and defined the canonical basis $\mathbf{B}(\Lambda(\boldsymbol{\lambda}))$ of the tensor product

$$\Lambda(\boldsymbol{\lambda}) = {}^{\omega}\Lambda_{\lambda_1} \otimes \cdots \otimes {}^{\omega}\Lambda_{\lambda_m} \otimes \Lambda_{\lambda_{m+1}} \otimes \cdots \otimes \Lambda_{\lambda_{m+n}}.$$

Lusztig's Positivity conjecture for tensor product

Conjecture (Lusztig)

For $\dot{b} \in \dot{\mathbf{B}}$, $\dot{b}\mathbf{B}(\Lambda(\boldsymbol{\lambda})) \subset \mathbb{N}[v, v^{-1}][\mathbf{B}(\Lambda(\boldsymbol{\lambda}))]$.

State-of-art:

- Finite and affine type A: based on works by Beilinson-Lusztig-MacPherson, Lusztig, Ginzburg, Schiffmann-Vasserot, Fu-Shoji:

$$\dot{b}(\mathbf{B}({}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}) \subset \mathbb{N}[v, v^{-1}][\mathbf{B}({}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2})] \text{ for } \dot{b} \in \dot{\mathbf{B}}.$$

- General type, highest weight modules only (Webster, Fang-Lan)

$$\mathbf{B}(\Lambda(\boldsymbol{\lambda})) \subset \mathbb{N}[v^{-1}][\mathbf{B}(\Lambda_{\lambda_1}) \otimes \cdots \otimes \mathbf{B}(\Lambda_{\lambda_n})];$$
$$E_i(\mathbf{B}(\Lambda(\boldsymbol{\lambda}))), F_i(\mathbf{B}(\Lambda(\boldsymbol{\lambda}))) \subset \mathbb{N}[v, v^{-1}][\mathbf{B}(\Lambda(\boldsymbol{\lambda}))] \text{ for } i \in I.$$

Lusztig's positivity conjecture for $\dot{\mathbf{U}}$

Conjecture (Lusztig)

1. *(Positivity with respect to the multiplication) For* $\dot{b}, \dot{b}' \in \dot{\mathbf{B}}$, $\dot{b}\dot{b}' \in \mathbb{N}[v, v^{-1}][\dot{\mathbf{B}}]$;
2. *(Positivity with respect to the comultiplication) For* $\dot{b} \in \dot{\mathbf{B}}$, $\Delta(\dot{b}) \in \mathbb{N}[v, v^{-1}]\dot{\mathbf{B}} \otimes \dot{\mathbf{B}}$.

State-of-art:

- Finite and affine type A: multiplication (Beilinson-Lusztig-MacPherson, Lusztig, Ginzburg, Schiffmann-Vasserot, Fu-Shoji); comultiplication (Fan-Li, Fu).
- Finite type: Webster claimed that $\dot{\mathbf{B}}$ has the positivity with respect to the multiplication.

These results are highly non-trivial and depend on geometric methods.

Main Result I: Comparison of structure constant

Theorem (Fang-H.)

For any dominant weights λ_1, λ_2, the entries of the transition matrix from the pure tensor basis $\mathbf{B}({}^{\omega}\Lambda_{\lambda_1}) \otimes \mathbf{B}(\Lambda_{\lambda_2})$ to the canonical basis $\mathbf{B}({}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2})$ of the tensor product ${}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}$ are given by the structure constants of the comultiplication in $\tilde{\mathbf{U}}^-$ with respect to its canonical basis $\tilde{\mathbf{B}}$.

Theorem (Fang-H.)

For any $\dot{b}, \dot{b}' \in \dot{\mathbf{B}}$, if $\dot{b}$ is spherical parabolic, then the structure constants of the multiplication $\dot{b}\dot{b}'$ in $\dot{\mathbf{U}}$ with respect to $\dot{\mathbf{B}}$ are given by the structure constants of the multiplication in $\tilde{\tilde{\mathbf{U}}}^-$ with respect to its canonical basis $\tilde{\tilde{\mathbf{B}}}$.

Main Result II: Positivity on the tensor product

Theorem (Fang-H.)

1. *(Positivity of the transition matrix)*

$$\mathbf{B}(\Lambda(\boldsymbol{\lambda})) \subset \mathbb{N}[v^{-1}][\mathbf{B}({}^{\omega}\Lambda_{\lambda_1}) \otimes \cdots \otimes \mathbf{B}(\Lambda_{\lambda_{m+n}})].$$

2. *(Positivity with respect to the action) For spherical parabolic $\dot{b}$ in the canonical basis $\dot{\mathrm{B}}$ of $\dot{\mathrm{U}}$,*

$$\dot{b}(\mathbf{B}(\Lambda(\boldsymbol{\lambda}))) \subset \mathbb{N}[v, v^{-1}][\mathbf{B}(\Lambda(\boldsymbol{\lambda}))].$$

3. *(Action on tensor product of highest weight modules) For $\dot{b} \in \dot{\mathbf{B}}$ and $\lambda_1, \lambda_n \in X^+$,*

$$\dot{b}(\mathbf{B}(\Lambda_{\lambda_1} \otimes \cdots \otimes \Lambda_{\lambda_n})) \subset \mathbb{N}[v, v^{-1}][\mathbf{B}(\Lambda_{\lambda_1} \otimes \cdots \otimes \Lambda_{\lambda_n})].$$

Main Result III: Positivity for $\dot{\mathbf{U}}$

The canonical bases $\mathbf{B}({}^{\omega}\Lambda_{\lambda_1+\lambda} \otimes \Lambda_{\lambda_2+\lambda})$ of ${}^{\omega}\Lambda_{\lambda_1+\lambda} \otimes \Lambda_{\lambda_2+\lambda}$ have very nice stability property when λ increases, and they can be glued together to form the canonical basis of the modified quantum group.

Theorem (Fang-H.)

For $\dot{b}, \dot{b}' \in \dot{\mathbf{B}}$, if one of them is spherical parabolic, then

$$\dot{b}\dot{b}' \in \mathbb{N}[v, v^{-1}][\dot{\mathbf{B}}];$$

Motivation: The Shadow of Positivity at q=1

- In 1930s Schoenberg, Gantmacher and Krein introduced the notion of totally positive real matrices: a matrix in $GL_n(\mathbb{R})$ is called totally positive (resp. totally nonnegative) if all its minors are positive (resp. nonnegative).
- In 1994, Lusztig introduced the theory of total positivity on Lie groups and their flag varieties using the theory of canonical basis.
- In some sense, the total positivity in geometry is the shadow (at $v = 1$) of a deeper quantum phenomenon: the algebraic positivity properties of the canonical basis.

Totally positive flags

Let G be a Kac-Moody group, split over $\mathbb{R}$ and $\mathcal{B} = G/B^+$ be the flag variety. Lusztig showed that for dominant regular weight λ, the following commuting diagram is Cartesian

$$\begin{tikzcd}
\mathcal{B}_{\geq 0} \arrow[r, hook] \arrow[d] & \mathbb{P}(\Lambda_\lambda)_{\geq 0} \arrow[d] \\
\mathcal{B} \arrow[r, hook] & \mathbb{P}(\Lambda_\lambda)
\end{tikzcd}$$

Theorem (Lusztig, Galashin-Karp-Lam, Bao-H., etc.)

The totally nonnegative flag $\mathcal{B}_{\geq 0}$ is a remarkable polyhedron space and the closure of each cell is homeomorphic to a closed ball.

Twisted product of flag varieties

- We consider the twisted product of flag varieties $X = G \times^{B^+} G/B^{\pm}$.
- It is a natural question to understand the totally nonnegative part of X. This would consequently implies the Fomin-Zelevinsky conjecture on double Bruhat cells.
- For many years, the only example we understand is for $G = GL_2$, where $X_{\geq 0} = \{(a, b); 0 \leq a \leq b \leq \infty\} \subsetneq \mathcal{B}_{\geq 0} \times \mathcal{B}_{\geq 0}$.
- This is consistent with the fact that the canonical basis of tensor product (conjecturally corresponding to $X_{\geq 0}$) is "more positive" than the tensor product of canonical basis (corresponding to $\mathcal{B}_{\geq 0} \times \mathcal{B}_{\geq 0}$).

Thickening method

The totally nonnegative part of $G \times^{B^+} G/B^+$ is finally understood in a joint work of Bao-H. via the thickening method.
It embeds the totally nonnegative part of the twisted product of $G \times^{B^+} G/B^+$ for a group G into the totally nonnegative part of a single flag variety for a larger Kac-Moody group $\tilde{G}$.
The key philosophy is that the enhanced symmetry of $\tilde{G}$ provides a powerful tool to unravel the intricate geometry of the twisted product (with, a priori, only the symmetry from G). We have a natural embedding

$$G \times^{B^+} G/B^+ \to \mathbb{P}(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}).$$

This geometric philosophy also works algebraically for tensor products. This is done by Fang-Lan for tensor product of highest weight modules and Fang-H. in the general case.

The Thickening Construction

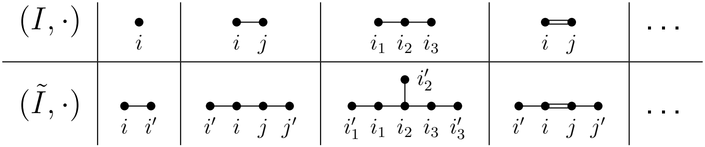
The *thickening Cartan datum* of $(I,\cdot)$ is $(\tilde{I},\cdot)$, where $\tilde{I} = I \sqcup I' := I \sqcup \{i'; i \in I\}$ and $\mathbb{Z}[\tilde{I}] \times \mathbb{Z}[\tilde{I}] \to \mathbb{Z}$ is extended from $\mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ in $(I,\cdot)$ such that

$$i \cdot j' = -\delta_{i,j}, \ i' \cdot j' = 2\delta_{i,j} \text{ for } i, j \in I.$$

In the language of generalized Cartan matrix:

$$(I,\cdot) \longleftrightarrow C = (i \cdot j)_{i,j \in I}, \quad (\tilde{I},\cdot) \longleftrightarrow \tilde{C} = \begin{pmatrix} C & -\mathrm{Id} \\ -\mathrm{Id} & 2\mathrm{Id} \end{pmatrix}.$$

In the language of Coxeter diagram:



Thickening product

- Let $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{U}}$ be the algebras corresponding to the thickening data $(\tilde{I}, \cdot)$.
- Let $\lambda \in X^+$. The element

$$\theta_\lambda = \prod_{i \in I} \theta_{i'}^{(\langle i, \lambda \rangle)} \in \tilde{\mathbf{f}}$$

  is independent of the order in the product.
- Let $\zeta \in X$ and $\lambda \in X^+$. The *thickening product* of ζ and λ is $\zeta \odot \lambda \in \tilde{X}$ defined by

$$\langle \mu, \zeta \odot \lambda \rangle = \langle \mu, \zeta + \lambda + |\theta_\lambda| \rangle \text{ for } \mu \in Y;$$
$$\langle i', \zeta \odot \lambda \rangle = \langle i, \lambda \rangle \text{ for } i \in I.$$

Strategy

- The simple lowest weight module ${}^{\omega}\Lambda_{\lambda_1}$ is the direct limit of its Demazure submodules $\varinjlim_{w\in W} {}^{\omega}V_w(\lambda_1)$.
- The tensor product ${}^{\omega}V_w(\lambda_1)\otimes\Lambda_{\lambda_2}$ is a quotient of the tensor product $M_{-w\lambda_1}\otimes\Lambda_{\lambda_2}$.
- Let $M_{\zeta\odot\lambda}$ be the Verma module of $\tilde{\mathbf{U}}$ with the highest weight $\zeta\odot\lambda\in\tilde{X}$. The subspace $(\mathbf{f}\theta_\lambda\mathbf{f})_{\zeta\odot\lambda}\subset\tilde{\mathbf{f}}=M_{\zeta\odot\lambda}$ spanned by $x\theta_\lambda y$ for $x,y\in\mathbf{f}$, is a $\mathbf{U}$-submodule of $M_{\zeta\odot\lambda}$.
- We define the $\mathbf{U}(\mathfrak{b}^-)$-module

$$\Lambda_{-w\lambda_1,\lambda_2}=(\mathbf{f}\theta_{\lambda_2}\mathbf{f})_{-w\lambda_1\odot\lambda_2}/((\mathbf{f}\theta_{\lambda_2}\mathbf{f})_{-w\lambda_1\odot\lambda_2}\cap\tilde{\mathbf{f}}\mathrm{Ann}_{\mathbf{f}}(\xi_{-w\lambda_1})).$$

Let $\pi_{-w\lambda_1,\lambda_2}:(\mathbf{f}\theta_{\lambda_2}\mathbf{f})_{-w\lambda_1\odot\lambda_2}\to\Lambda_{-w\lambda_1,\lambda_2}$ be the natural projection.

Outline

We introduce the following thickening construction

$$\begin{tikzcd}
\mathbf{B}((\mathbf{f}\theta_{\lambda_2}\mathbf{f})_{-w\lambda_1\odot\lambda_2}) \arrow[r, shift left, ->>] \arrow[d, hook] & \mathbf{B}(M_{-w\lambda_1}\otimes\Lambda_{\lambda_2}) \arrow[l, shift left] \arrow[r, ->>] \arrow[d, hook] & \mathbf{B}({}^{\omega}V_w(\lambda_1)\otimes\Lambda_{\lambda_2})\sqcup\{0\} \arrow[d, hook] \\
(\mathbf{f}\theta_{\lambda_2}\mathbf{f})_{-w\lambda_1\odot\lambda_2} \arrow[r, shift left, ->>, "\varphi_{-w\lambda_1,\lambda_2}"] & M_{-w\lambda_1}\otimes\Lambda_{\lambda_2} \arrow[l, shift left, "\psi_{-w\lambda_1,\lambda_2}"] \arrow[r, ->>, "a\otimes\mathrm{Id}"] & {}^{\omega}V_w(\lambda_1)\otimes\Lambda_{\lambda_2}.
\end{tikzcd}$$

to embed $M_{-w\lambda_1}\otimes\Lambda_{\lambda_2}$ into $\tilde{\mathbf{U}}^-$ of a larger quantum group $\tilde{\mathbf{U}}$. This allows us to transfer the (known) positivity in $\tilde{\mathbf{U}}^-$ to the (desired) positivity in the tensor product.

A Concrete Example for GL_2

- Let $(I, \cdot)$ be of type GL_2. Its thickening Cartan datum is of type GL_3.
- The canonical basis of $\mathbf{f}$ is $\mathbf{B} = \{\theta_i^{(k)}; k \in \mathbb{N}\}$.
- For dominant weights $m, n \in \mathbb{N}$,

  $\mathbf{B}({}^{\omega}\Lambda_m) = \{E_i^{(k)}\xi_{-m}; 0 \leqslant k \leqslant m\}$, $\mathbf{B}(\Lambda_n) = \{F_i^{(l)}\eta_n; 0 \leqslant l \leqslant n\}$.
- The canonical basis of ${}^{\omega}\Lambda_m \otimes \Lambda_n$ is

$$\mathbf{B}({}^{\omega}\Lambda_m \otimes \Lambda_n) = \{E_i^{(k)}\xi_{-m}\diamondsuit F_i^{(l)}\eta_n; 0 \leqslant k \leqslant m, 0 \leqslant l \leqslant n\},$$

  where $E_i^{(k)}\xi_{-m}\diamondsuit F_i^{(l)}\eta_n =$

$$\begin{cases} \sum\limits_{s=0}^{\min(k,l)} v^{s(k-m-s)} \frac{[n-l+s]!}{[s]![n-l]!} E_i^{(k-s)}\xi_{-m} \otimes F_i^{(l-s)}\eta_n, & \text{if } k-l \leqslant m-n; \\ \sum\limits_{s=0}^{\min(k,l)} v^{s(l-n-s)} \frac{[m-k+s]!}{[s]![m-k]!} E_i^{(k-s)}\xi_{-m} \otimes F_i^{(l-s)}\eta_n, & \text{if } k-l \geqslant m-n. \end{cases}$$

Example

- The canonical basis of $\tilde{\mathbf{f}}$ is

$$\tilde{\mathbf{B}} = \{\theta_i^{(p)}\theta_{i'}^{(q)}\theta_i^{(r)}, \theta_{i'}^{(r)}\theta_i^{(q)}\theta_{i'}^{(p)}; p, q, r \in \mathbb{N}, q \geqslant p + r\}$$

with the identification $\theta_i^{(p)}\theta_{i'}^{(q)}\theta_i^{(r)} = \theta_{i'}^{(r)}\theta_i^{(q)}\theta_{i'}^{(p)}$ when $q = p + r$.

- We have $E_i^{(k)}\xi_{-m}\diamondsuit F_i^{(l)}\eta_n =$
$$\begin{cases} \underline{\varphi_{m,n}}\pi_{m,n}(\theta_{i'}^{(n-l)}\theta_i^{(m-k+l)}\theta_{i'}^{(l)}), & \text{if } k - l \leqslant m - n; \\ \underline{\varphi_{m,n}}\pi_{m,n}(\theta_i^{(l)}\theta_{i'}^{(n)}\theta_i^{(m-k)}), & \text{if } k - l \geqslant m - n. \end{cases}$$

Synthesis: Bridging Algebra and Geometry

We have the following commutative diagrams (geometric side by Bao-H., H.-Xie, algebraic side by Fang-Lan, Fang-H.)

$$\begin{tikzcd}
\text{[Bao-H.]}(G \times^{B^+} G/B^+)_{\geq 0} \arrow[r, hook] \arrow[d] & \mathbb{P}(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2})_{\geq 0}\text{[Fang-Lan]} \arrow[d] \\
G \times^{B^+} G/B^+ \arrow[r, hook] & \mathbb{P}(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2})
\end{tikzcd}$$

$$\begin{tikzcd}
\text{[H.-Xie]}(G \times^{B^+} G/B^-)_{\geq 0} \arrow[r, hook] \arrow[d] & \mathbb{P}({}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2})_{\geq 0}\text{[Fang-H.]} \arrow[d] \\
G \times^{B^+} G/B^- \arrow[r, hook] & \mathbb{P}({}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2})
\end{tikzcd}$$

Open Problem: Can geometric positivity be completely detected by algebraic positivity? That is, are these diagrams Cartesian?

Happy birthday, Michel!